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On elliptical billiards in the Lobachevsky space and associated geodesic hierarchies

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Abstract

We derive Cayley's type conditions for periodical trajectories for the billiard within an ellipsoid in the Lobachevsky space. It appears that these new conditions are of the same form as those obtained before for the Euclidean case. We explain this coincidence by using theory of geodesically equivalent metrics and show that Lobachevsky and Euclidean elliptic billiards can be naturally considered as a part of a hierarchy of integrable elliptical billiards.

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1. Introduction

We start with the following well-known integrable mechanical system: motion of a free particle within an ellipsoid in the Euclidean space of any dimension d. On the boundary, the particle obeys the billiard law. Integrability of the system is related to classical geometrical properties of elliptical billiards: the Chasles, Poncelet and Cayley theorems. According to the Chasles theorem [1] every line in this space is tangent to $d-1$ quadrics confocal to the outer ellipsoid. Even more, all segments of the particle's trajectory are tangent to the same $d-1$ quadrics [26]. The Poncelet theorem [13,22,28] put some light on closed billiard trajectories: *there exists a closed trajectory with $d-1$ given confocal caustics if and only*

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if infinitely many such trajectories exist, and all of them have the same period. Since the periodicity of a billiard trajectory depends only on its caustic surfaces, it is a natural question to find an analytical connection between them and corresponding period.

The Poncelet theorem, as one of the highlights of the 19th century projective geometry, attracted the attention of Arthur Cayley for several years (see [7–12]). In [8], Cayley found the analytical condition for caustic conics in the Euclidean plane case. The classical and algebro-geometric proofs of Cayley's theorem can be found in Lebesgue's book [28] and Griffiths and Harris paper [23], respectively. The generalisation is established by Dragović and Radnović for any d [18,19]. This generalisation was done by use of the Veselov–Moser discrete quadratic $L - A$ pair for the classical Heisenberg magnetic model [32]. The integrability of elliptical billiard systems in the Lobachevsky space was proved by Veselov in [38]. There, Veselov used discrete linear $L - A$ pair, which is quite different from the one used in the Euclidean case. The starting point of this paper is derivation of Cayley's type conditions for the Lobachevsky billiard and our observation that these new conditions coincide with those obtained in [18,19] for the Euclidean case (Section 3). We found a natural way to explain this coincidence and it is related to the recently developed integrability approach in the theory of geodesically equivalent metrics [29,35]. Both Lobachevsky and Euclidean elliptic billiards can be naturally considered as members of a hierarchy of integrable elliptical billiards (Section 4). In the conclusion of this section, we present some properties of the Laurent polynomial integrable potential perturbations of those separable systems, continuing the study of such systems which started with [14], see also [15–17,24].

2. Basic notions on billiard systems

Let (Q, g) be a d-dimensional Riemannian manifold and let $D \subset Q$ be a domain with a smooth boundary Γ. Let $\pi : T^*Q \to Q$ be a natural projection and let g^{-1} be the contravariant metric on the cotangent bundle, in coordinates

$$|p| = \sqrt{g^{-1}(p, p)} = \sqrt{g^{ij} p_i p_j}, \quad p \in T_x^* Q.$$

Consider the *reflection mapping*

$$r : \pi^{-1}\Gamma \to \pi^{-1}\Gamma, \qquad p_- \mapsto p_+,$$

which associates the covector $p_+ \in T_x^*Q$, $x \in \Gamma$ to a covector $p_- \in T_x^*Q$ such that the following conditions hold:

$$|p_+| = |p_-|, \qquad p_+ - p_- \perp \Gamma. \tag{1}$$

A *billiard* in D is a dynamical system with the phase space $M = T^*D$ whose trajectories are geodesics given by the Hamiltonian equations

$$\dot{p} = -\frac{\partial H}{\partial x}, \qquad \dot{x} = \frac{\partial H}{\partial p}, \qquad H(p, x) = \frac{1}{2} g_x^{-1}(p, p), \tag{2}$$

reflected at points $x \in \Gamma$ according to the billiard law: $r(p_-) = p_+$. Here p_- and p_+ denote the momenta before and after the reflection. If some potential force field $V(x)$ is added than the system is described with the same reflection law (1) and Hamiltonian equations (2) with the Hamiltonian $H(p, x) = (1/2)g_x^{-1}(p, p) + V(x)$. A function $f : T^*Q \to \mathbb{R}$ is an *integral* of the billiard system if it commutes with the Hamiltonian ($\{f, H\} = 0$) and does not change under the reflection ($f(x, p) = f(x, r(p)), x \in \Gamma$). The billiard is *completely integrable in the sense of Birkhoff* if it has d integrals polynomial in the momenta, which are in involution, and almost everywhere independent (see [26]). The classical integrable examples, with smooth boundary, are billiards inside ellipsoids on the Euclidean and hyperbolic spaces and spheres, with integrals quadratic in the velocities [26]. These systems can be also considered as discrete integrable systems [37,38]. The explicit integrations in terms of theta-functions are performed by Veselov [37,38], Moser and Veselov [32], and Fedorov [20].

3. Poncelet theorem and Cayley's condition for the billiard in the Lobachevsky space

Veselov proved the integrability of the billiard system within an ellipsoid in the Lobachevsky space in [38]. He showed that its motion corresponds to certain translations of the Jacobi variety of some hyperelliptic curve and gave explicit formulae of the motion in terms of theta-functions. The aim of this section is to find an analogue of Poncelet's and Cayley's theorem [8] for the billiard motion within an ellipsoid in the Lobachevsky space.

3.1. Integration of the billiard motion in the Lobachevsky space: Poncelet theorem

For a brief account of Veselov's results on the billiard in the Lobachevsky space [38], let us consider the $(d + 1)$-dimensional Minkowski space $V = \mathbb{R}^{d,1}$ with the symmetric bilinear form:

$$\langle \xi, \eta \rangle = -\xi_0\eta_0 + \xi_1\eta_1 + \cdots + \xi_d\eta_d.$$

One sheet of the hyperboloid $\langle \xi, \xi \rangle = -1$ with the induced metric is a model of the d-dimensional Lobachevsky space $\mathbb{H}^d$. An ellipsoid Γ in this space is determined by the equation

$$\Gamma = \left\{ \xi \in \mathbb{H}^d, -\frac{\xi_0^2}{a_0} + \frac{\xi_1^2}{a_1} + \cdots + \frac{\xi_d^2}{a_d} = 0 \right\}, \tag{3}$$

with $a_0 > a_1 \geq a_2 \geq \cdots \geq a_d > 0$. All segments of the billiard trajectory within this ellipsoid are tangent to $d - 1$ confocal quadric surfaces (including multiplicity), fixed for a given trajectory (Theorem 3 in [38]). Denote by μ_i, $i = 1, \ldots, d - 1$ the numbers such that the equations of these caustics are:

$$-\frac{x_0^2}{a_0 - \mu_i} + \frac{x_1^2}{a_1 - \mu_i} + \cdots + \frac{x_d^2}{a_d - \mu_i} = 0 \quad (1 \leq i \leq d - 1). \tag{4}$$

Then the points of reflection from the boundary Γ correspond to the shift $D_{k+1} = D_k + Q_- - Q_+$ on the Jacobi variety of the spectral curve $\mathcal{C}$

$$\mathcal{C} : (\mu - a_0) \cdots (\mu - a_d) = c \cdot \lambda^2(\mu - \mu_1) \cdots (\mu - \mu_{d-1}), \tag{5}$$

where c is a constant, and Q_+, Q_- are the points on the curve $\mathcal{C}$ over $\mu = 0$. (See Theorem 2 of [38]. The curve $\mathcal{C}$ is the spectral curve of the $L - A$ pair considered there.) Let us note that Veselov considered only the case of the regular (hyperelliptic) curve $\mathcal{C}$ [38]. However, his consideration holds for the singular case, too. Suppose a periodical billiard trajectory inside the ellipsoid Γ in the Lobachevsky space is given. All trajectories with the same caustics have the same spectral curve. If the period of the given trajectory is n, then $n(Q_+ - Q_-) = 0$ on $\mathrm{Jac}(\mathcal{C})$, and vice versa. Thus, all these trajectories close after n bounces. Therefore, Poncelet's-type theorem for the billiard in the Lobachevsky space is derived from Veselov's results.

Proposition 1. *Suppose a periodical billiard trajectory inside an ellipsoid in the Lobachevsky space is given. Then any billiard trajectory which shares the same caustic quadrics is also periodical, with the same period.*

3.2. Cayley's conditions—regular spectral curve

Assume that all constants $a_0, a_1, \dots, a_d, \mu_1, \dots, \mu_{d-1}$ are mutually different. Then the spectral curve $\mathcal{C}$ is hyperelliptic. Cases when some of them coincide are discussed in the next subsection. To establish an analytical condition on a trajectory to be periodic with period n, we need to find out when the divisors nQ_+ and nQ_- on the spectral curve are equivalent.

Lemma 1. *Let the curve C be given by*

$$y^2 = (x - x_1)\cdots(x - x_{2g+2}), \tag{6}$$

with all x_i mutually different and not equal to 0, *and Q_+, Q_- the two points on C over the point $x = 0$. Then $nQ_+ \equiv nQ_-$ is equivalent to*

$$\operatorname{rank}\begin{bmatrix} B_{g+2} & B_{g+3} & \cdots & B_{n+1} \\ B_{g+3} & B_{g+4} & \cdots & B_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ B_{g+n} & \cdots & \cdots & B_{2n-1} \end{bmatrix} < n - g \quad \textit{and} \quad n > g, \tag{7}$$

where $y = \sqrt{(x - x_1)\cdots(x - x_{2g+2})} = B_0 + B_1 x + B_2 x^2 + \cdots$ is the Taylor expansion around the point Q_-.

Proof. C is a hyperelliptic curve of genus g. The relation $nQ_+ \equiv nQ_-$ means that there exists a meromorphic function on C with a pole of order n at the point Q_+, a zero of the same order at Q_- and neither other zeros nor poles. Denote by $L(nQ_+)$ the vector space of meromorphic functions on C with a unique pole Q_+ of order at most n. Since Q_+ is not a branching point on the curve, $\dim L(nQ_+) = 1$ for $n \le g$, and $\dim L(nQ_+) = n - g + 1$, for $n > g$. In the case $n \le g$, the space $L(nQ_+)$ contains only constant functions, and the divisors nQ_+ and nQ_- cannot be equivalent. If $n \ge g + 1$, we choose the following basis for $L(nQ_+)$:

$$1, f_1, \dots, f_{n-g},$$

where

$$f_k = \frac{y - B_0 - B_1 x - \cdots - B_{g+k-1} x^{g+k-1}}{x^{g+k}}.$$

Thus, $nQ_+ \equiv nQ_-$ if there is a function $f \in L(nQ_+)$ with a zero of order n at Q_-, i.e., if there exist constants $\alpha_0, \dots, \alpha_{n-g}$, not all equal to 0, such that

$$\begin{aligned} &\alpha_0 + \alpha_1 f_1(Q_-) + \cdots \alpha_{n-g} f_{n-g}(Q_-) = 0,\\ &\alpha_1 f_1'(Q_-) + \cdots \alpha_{n-g} f_{n-g}'(Q_-) = 0,\\ &\vdots\\ &\alpha_1 f_1^{(n-1)}(Q_-) + \cdots \alpha_{n-g} f_{n-g}^{(n-1)}(Q_-) = 0. \end{aligned}$$

Existence of a non-trivial solution to this system of linear equations is equivalent to the condition (7). □

Introducing new coordinates $x = \mu$, $y = \sqrt{c}\lambda(\mu - \mu_1)\cdots(\mu - \mu_{d-1})$, the spectral curve (5) is transformed to

$$y^2 = (x - a_0)\cdots(x - a_d)(x - \mu_1)\cdots(x - \mu_{d-1}), \tag{8}$$

and we obtain the following theorem.

Theorem 1. *The condition of a billiard trajectory inside the ellipsoid* (3) *in the d-dimensional Lobachevsky space, with non-degenerate caustics* (4), *to be periodic with period* $n \geq d$ *is*

$$\operatorname{rank}\begin{bmatrix} B_{n+1} & B_n & \cdots & B_{d+1}\\ B_{n+2} & B_{n+1} & \cdots & B_{d+2}\\ \cdots & \cdots & \cdots & \cdots\\ B_{2n-1} & B_{2n-2} & \cdots & B_{n+d-1} \end{bmatrix} < n - d + 1,$$

where $\sqrt{(x - a_0)\cdots(x - a_d)(x - \mu_1)\cdots(x - \mu_{d-1})} = B_0 + B_1 x + B_2 x^2 + \cdots$. *There is no such trajectories with period less than* d.

3.3. Cases of singular spectral curve

When all $a_0, a_1, \dots, a_d, \mu_1, \dots, \mu_{d-1}$ are mutually different, then the curve (5) has no singularities in the affine part. However, singularities appear in the following three cases and their combinations:

(i) $a_i = \mu_j$ for some i, j. The spectral curve (5) decomposes into a rational and a hyperelliptic curve. Geometrically, this means that the caustic corresponding to μ_i degenerates into hyper-plane $x_i = 0$. The billiard trajectory can be asymptotically tending to that hyper-plane (and therefore cannot be periodic), or completely placed in this hyper-plane. Therefore, the closed trajectories appear when they are placed in a coordinate hyper-plane. Such motion can be discussed like in the case of dimension $d - 1$.

(ii) $a_i = a_j$ for some $i \neq j$. The ellipsoid (3) is symmetric.
(iii) $\mu_i = \mu_j$ for some $i \neq j$. The billiard trajectory is placed on the corresponding confocal quadric hyper-surface.[1]

In the cases (ii) and (iii) the spectral curve $\mathcal{C}$ is a hyperelliptic curve with singularities. In spite of their different geometrical nature, they both need the same analysis of the condition $nQ_+ \equiv nQ_-$ for the singular curve (5).

Lemma 2. *Let the curve C be given by $y^2 = (x - x_1) \cdots (x - x_{2g+2})$, with all x_i different from* 0, *and Q_+, Q_- the two points on C over the point $x = 0$. Then $nQ_+ \equiv nQ_-$ is equivalent to* (7) *where $y = \sqrt{(x - x_1) \cdots (x - x_{2g+2})} = B_0 + B_1 x + B_2 x^2 + \cdots$ is the Taylor expansion around the point Q_-.*

Proof. Suppose that, among $x_1, \dots, x_{2g+2}$, only x_{2g+1} and x_{2g+2} have same values. Then $(x_{2g+1}, 0)$ is an ordinary double point on C. The normalisation of the curve C is the pair $(\tilde{C}, \pi)$, where $\tilde{C}$ is the curve given by

$$\tilde{C} : \tilde{y}^2 = (\tilde{x} - x_1) \cdots (\tilde{x} - x_{2g}),$$

and $\pi : \tilde{C} \to C$ is the projection:

$$(\tilde{x}, \tilde{y}) \overset{\pi}{\mapsto} (x = \tilde{x}, y = (\tilde{x} - x_{2g+1})\tilde{y}).$$

The genus of $\tilde{C}$ is $g-1$. The relation $nQ_+ \equiv nQ_-$ is equivalent to existence of a meromorphic function f on $\tilde{C}$, $f \in L(n\tilde{Q}_+)$, with a zero of order n at $\tilde{Q}_-$, and $f(A) = f(B)$, where $\tilde{Q}_+$, $\tilde{Q}_-$ are the two points over $\tilde{x} = 0$, and A, B are over $\tilde{x} = x_{2g+1}$. For $n \leq g - 1$, $\dim L(n\tilde{Q}_+) = 1$, and this space contains only constant functions. For $n \geq g$, we can choose the following basis for $L(n\tilde{Q}_+)$:

$$1,\ f_0,\ f_1 \circ \pi, \dots, f_{n-g} \circ \pi.$$

f_k are as in Lemma 1 for $k > 0$, and

$$f_0 = \frac{\tilde{y} - \tilde{B}_0 - \tilde{B}_1 \tilde{x} - \cdots - \tilde{B}_{g-1} \tilde{x}^{g-1}}{\tilde{x}^g},$$

where $\tilde{y} = \sqrt{(\tilde{x} - x_1) \cdots (\tilde{x} - x_{2g})} = \tilde{B}_0 + \tilde{B}_1 \tilde{x} + \tilde{B}_2 \tilde{x}^2 + \cdots$ is the Taylor expansion around the point $\tilde{Q}_-$. Since f_0 is the only element of the basis with different values in the points A and B, we obtain that $n\tilde{Q}_+ \equiv n\tilde{Q}_-$ is equivalent to (7). Cases when C has more singularities, or singular points of higher order, can be discussed in the similar manner. $\square$

Immediate consequence of Lemma 2 is that Theorem 1 can be applied not only for the case of the non-singular spectral curve, but in the cases (ii) and (iii) too. Therefore, the following interesting property holds.

[1] We learned about its geometric significance from Prof. Yurij Fedorov.

Theorem 2. *If the billiard trajectory within an ellipsoid Γ in d-dimensional Lobachevsky space is periodic with period $n < d$, then it is placed in one of the n-dimensional planes of symmetry of the ellipsoid.*

This property can be seen easily for $d = 3$.

Example 1. Consider the billiard motion in an ellipsoid in the three-dimensional space, with $\mu_1 = \mu_2$, when the segments of the trajectory are placed on generatrices of the corresponding quadric surface confocal to the ellipsoid. If there existed a periodic trajectory with period $n = d = 3$, the three bounces would have been coplanar, and the intersection of that plane and the quadric would have consisted of three lines, which is impossible. It is obvious that any periodic trajectory with period $n = 2$ is placed along one of the axes of the ellipsoid. So, there is no periodic trajectories contained in a confocal quadric surface, with period less or equal to 3.

4. Hierarchy of integrable elliptical billiards

4.1. The Beltrami–Klein model of the Lobachevsky space

Note first that Cayley's type conditions for the Lobachevsky billiard from Section 3 are of the same form as those obtained in [18,19] for the Euclidean case, although the $L - A$ pairs used there are quite different. There is a natural way to explain this coincidence. We use the Beltrami–Klein model of the Lobachevsky space $\mathbb{H}^d$. The coordinate transformation

$$y_1 = \frac{\xi_1}{\xi_0}, \dots, y_d = \frac{\xi_d}{\xi_0}$$

maps the Lobachevsky space, modeled as a pseudosphere of the Minkowski space, to the Beltrami–Klein model within the unit sphere in $\mathbb{R}^d$ [38]. Now, after appropriate linear changing of coordinates $x_1 = \alpha_1 y_1, \dots, x_d = \alpha_d y_d$ we can obtain the Beltrami–Klein model inside the ellipsoid Λ:

$$\Lambda = \left\{ x = (x_1, \dots, x_d) \in \mathbb{R}^d, \frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} + \cdots + \frac{x_d^2}{b_d} = 1 \right\},$$

such that the ellipsoid (3) in new coordinates is confocal to Λ. Then its equation can be written in the form:

$$\Gamma = \left\{ x \in \mathbb{R}^d, \frac{x_1^2}{b_1 - c} + \frac{x_2^2}{b_2 - c} + \cdots + \frac{x_d^2}{b_d - c} = 1 \right\},$$

where $0 < c < b_i$, $i = 1, \dots, d$. The hyperbolic metric within Λ is given by (for example, see [34]):

$$\mathrm{d}\bar{g}^2 = \frac{1}{b_1 \cdot b_2 \cdots b_d \cdot f^2}\left(f\left(\frac{\mathrm{d}x_1^2}{b_1} + \cdots + \frac{\mathrm{d}x_d^2}{b_d}\right) + \left(\frac{x_1\,\mathrm{d}x_1}{b_1} + \cdots + \frac{x_d\,\mathrm{d}x_d}{b_d}\right)^2\right),$$
$$f = 1 - \left(\frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} + \cdots + \frac{x_d^2}{b_d}\right).$$

The metric $\mathrm{d}\bar{g}^2$ can be written in the matrix form as $\mathrm{d}\bar{g}^2 = \langle \Pi\,\mathrm{d}x, \mathrm{d}x\rangle$, where $\mathrm{d}x = (\mathrm{d}x_1, \dots, \mathrm{d}x_d)$,

$$\Pi = \frac{1}{\det B \cdot f^2}(fB^{-1} + B^{-1}x \otimes B^{-1}x),$$

$B = \mathrm{diag}(b_1, \dots, b_d)$, and $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product. The hyperbolic metric has the same geodesics, considered as unparametrised curves, as the Euclidean metric $\mathrm{d}g^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_d^2$. Suppose that $b_1 > b_2 > \cdots > b_d$. The standard elliptic coordinates $\lambda_1, \dots, \lambda_d$ in $\mathbb{R}^d$ ($b_1 > \lambda_1 > b_2 > \lambda_2 > \cdots > \lambda_{d-1} > b_d > \lambda_d$) are defined as solutions of the equation:

$$\gamma(\lambda) = \frac{x_1^2}{b_1 - \lambda} + \frac{x_2^2}{b_2 - \lambda} + \cdots + \frac{x_d^2}{b_d - \lambda} = 1.$$

The direct verification shows that the metric $\mathrm{d}\bar{g}^2$, as well as the Euclidean metric $\mathrm{d}g^2$, is orthogonally separable in the elliptic coordinates and geodesic flows can be integrated by the theorem of Stäckel. This means that hypersurfaces $\lambda_i = \text{const.}$ of the coordinate system $\lambda_1, \dots, \lambda_d$ are orthogonal to each other and the corresponding Hamilton–Jacobi equations for the Hamiltonian of the geodesic flows have complete solutions of the form $\mathcal{S}(\lambda_1, \dots, \lambda_d, c_1, \dots, c_d) = \mathcal{S}_1(\lambda_1, c_1) + \cdots + \mathcal{S}_n(\lambda_d, c_d)$ (see [2,3] and references therein). Consider the billiard in the domain D bounded by the ellipsoid Γ. In elliptic coordinates, the boundary of the ellipsoid is given by the equation $\lambda_d = c$ and the reflection map, both for the Euclidean and Lobachevsky metrics, is given by

$$\begin{aligned}&(\lambda_1, \lambda_2, \dots, \lambda_{d-1}, c, p_{\lambda_1}, p_{\lambda_2}, \dots, p_{\lambda_{d-1}}, p_{\lambda_d})\\ &\quad\mapsto (\lambda_1, \lambda_2, \dots, \lambda_{d-1}, c, p_{\lambda_1}, p_{\lambda_2}, \dots, p_{\lambda_{d-1}}, -p_{\lambda_d}),\end{aligned} \tag{9}$$

where (λ, p_λ) are canonical coordinates in $T^*\mathbb{R}^d$. Such a simple form of the reflection map is due to the fact that Γ and Λ are confocal in the coordinates $x_1, \dots, x_d$. Therefore we have the following lemma.

Lemma 3. *The billiards inside the ellipsoid Γ in the Euclidean and the Lobachevsky space, modeled within the ellipsoid Λ, have the same trajectories up to reparametrisation.*

Lemma 3 provides the explanation for the coincidence of the Cayley's conditions obtained in the previous section and papers [18,19]. The above observation allows us to approach to the problem of the integrability of elliptical billiards in a new way, using theory of geodesically equivalent metrics.

4.2. Geodesically equivalent metrics

Let g and $\bar{g}$ be Riemannian metrics on d-dimensional manifold Q. The metrics g and $\bar{g}$ are called *geodesically equivalent* if they have the same geodesics considered as unparametrised curves. This is a classical subject studied by Beltrami, Dini, Levi-Civita, etc. in 19th century. Recently, the new global understanding of the theory is developed in the framework of integrable systems (see [5,29,35,36] and references therein). Having the metrics g and $\bar{g}$, define the $(1, 1)$-tensor field $L = L(g, \bar{g})$ by

$$L = \left(\frac{\det(\bar{g})}{\det(g)}\right)^{1/(d+1)} \bar{g}^{-1} g.$$

Consider functions

$$J_l(p, x) = g_x^{-1}(S_l p, p) = \sum_{j,k,i} (S_l)^i_j g^{jk} p_i p_k, \tag{10}$$

where $(1, 1)$ tensors S_k are given by the following formula:

$$\{\det(L + \alpha\,\mathrm{Id})\}(L + \alpha\,\mathrm{Id})^{-1} = S_{d-1}\alpha^{d-1} + S_{d-2}\alpha^{d-2} + \cdots + S_0, \quad \alpha \in \mathbb{R}.$$

If the metrics g and $\bar{g}$ are geodesically equivalent then functions $J_l(p, x)$ are in involution with respect to the canonical symplectic structure on T^*Q. Moreover, if the eigenvalues of L are all different at one point of Q, then they are different almost everywhere and the geodesic flows of g and $\bar{g}$ are completely integrable. The complete set of involutive integrals for the first flow is $J_0, J_1, \dots, J_{d-1}$ (see [29,35,36]). In this case we say that g and $\bar{g}$ are *strictly non-proportional*. The pair g, $\bar{g}$ of geodesically equivalent Riemannian metrics produces the family of geodesically equivalent Riemannian metrics g_k, $\bar{g}_k$, $k \in \mathbb{Z}$ given by the following formulas [35,36]:

$$g_k(\xi, \eta) = g(L^k \xi, \eta), \qquad \bar{g}_k(\xi, \eta) = g(L^k \xi, \eta), \quad k \in \mathbb{Z}. \tag{11}$$

The integrals of the geodesic flows of metric g_k and $\bar{g}_{k+2}$ canonically given by (10) coincide [35]. Following [5] we call (11) *Topalov–Sinjukov hierarchy* of Riemannian metrics on Q. The nice geometrical interpretation of geodesical equivalence is done by Bolsinov and Matveev [5]. They proved that g and $\bar{g}$ are geodesically equivalent if and only if L is a Benenti tensor field for the metric g [3,5]. This implies that if g and $\bar{g}$ are strictly non-proportional than all metrics g_k, $\bar{g}_k$ are orthogonally separable in the same coordinates and geodesic flows can be integrated by the theorem of Stäckel [5].

4.3. Hierarchy of integrable elliptical billiards

Now we shall apply the general construction described in the previous section to the Euclidean $\mathrm{d}g^2$ and Lobachevsky metrics $\mathrm{d}\bar{g}^2$ inside the ellipsoid Λ. Note that this natural geodesical equivalence is a slight modification of the geodesical equivalence studied by Topalov in [36]. Taking $b = B$, $\bar{a} = \sqrt{-1}x$, from Lemma 7 of [36] we are getting that $(1, 1)$ tensor field L has the following matrix form:

$$L = L(\mathrm{d}g^2, \mathrm{d}\bar{g}^2) = (\det \Pi)^{1/(d+1)} \Pi^{-1} = B - x \otimes x.$$

Therefore we have the Topalov–Sinjukov hierarchy of strictly non-proportional Riemannian metrics within Λ given by

$$\mathrm{d}g_k^2 = \langle (B - x \otimes x)^k \,\mathrm{d}x, \mathrm{d}x\rangle, \qquad \mathrm{d}\bar{g}_k^2 = \langle \Pi(B - x \otimes x)^k \,\mathrm{d}x, \mathrm{d}x\rangle. \tag{12}$$

The corresponding geodesic flows are completely integrable. The integrals of the geodesic flow of the metrics $\mathrm{d}g_k^2$ and $\mathrm{d}\bar{g}_{k+2}^2$ are:

$$J_i^k(p, x) = \langle S_i (B - x \otimes x)^{-k} p, p\rangle, \tag{13}$$

where

$$\begin{aligned} &\{\det(B - x \otimes x + \alpha\,\mathrm{Id})\}(B - x \otimes x + \alpha\,\mathrm{Id})^{-1} \\ &\quad = \det B_\alpha((1 - \langle B_\alpha^{-1} x, x\rangle) B_\alpha^{-1} + B_\alpha^{-1} \otimes B_\alpha^{-1}) = S_{d-1}\alpha^{d-1} + \cdots + S_0, \end{aligned} \tag{14}$$

$p = (p_1, \dots, p_d) \in T_x^*\mathbb{R}^d$ is the canonical momentum in Euclidean coordinates $x = (x_1, \dots, x_d)$, $B_\alpha = \mathrm{diag}(b_1 + \alpha, \dots, b_d + \alpha)$ and α is a real parameter. For $k = 0$ these functions are defined on the whole $T^*\mathbb{R}^d$ and coincide with commuting functions given by Moser in [31]. Consider the billiards in the domain D bounded by the ellipsoid Γ with metrics (12). According to [5] all metrics (12) are orthogonally separable in elliptical coordinates. This implies that the reflection map is the same for all metrics and in elliptic coordinates has the form (9). Moreover, since integrals $J_i^k(p, x)$ are diagonal in elliptic coordinates, we have that they are not just integrals of the geodesic flows of metrics $\mathrm{d}g_k^2$ and $\mathrm{d}\bar{g}_{k+2}^2$, but also integrals of the corresponding billiard systems inside ellipsoid Γ (see [26, pp. 133–134]). Thus we get the following general statement.

Theorem 3. *The billiard systems inside ellipsoid Γ with the Riemannian metrics $\mathrm{d}g_k^2$, $\mathrm{d}\bar{g}_k^2$ given by (12) are completely integrable for all $k \in \mathbb{Z}$. In particular, the elliptical billiards in the Euclidean and hyperbolic spaces are completely integrable.*

Since the reflection map r is the same for the whole hierarchy, applying Proposition 3 of [35], we get the following corollary.

Corollary 1. *The billiard systems for the given hierarchy have isomorphic Liouville foliations of T^*D/r.*

Remark 1. Matveev and Topalov [29] and Tabachnikov [34] proved that ellipsoid

$$\tilde{\Lambda} = \left\{ x = (x_1, x_2, \dots, x_{d+1}) \in \mathbb{R}^{d+1}, \frac{x_1^2}{b_1} + \frac{x_2^2}{b_2} + \cdots + \frac{x_{d+1}^2}{b_{d+1}} = 1 \right\}$$

admits non-trivial geodesic equivalence between the standard metric and the metric

$$\frac{1}{b_1 \cdot b_2 \cdots b_{d+1} \cdot ((x_1^2/b_1) + (x_2^2/b_2) + \cdots + (x_{d+1}^2/b_{d+1}))} \times \left(\frac{\mathrm{d}x_1^2}{b_1} + \frac{\mathrm{d}x_2^2}{b_2} + \cdots + \frac{\mathrm{d}x_{d+1}^2}{b_{d+1}} \right)\Bigg|_{\tilde{\Lambda}}.$$

The Euclidean and the Lobachevsky metrics within ellipsoid Λ can be seen as limits of the given metrics as b_{d+1} tends to zero.

4.4. *Integrable potential perturbations*

We shall say that the potential $V(x)$ is *separable in the elliptic coordinates* $\lambda_1, \dots, \lambda_d$ if the Hamilton–Jacobi equation for the Hamiltonian $(1/2)(p_1^2 + \cdots + p_d^2) + V(x)$ can be solved by separation of variables in elliptic coordinates. This definition, in a more geometrical fashion, can be found in [2,3]. The potential of the elastic force is an example. The potential $V(x)$ is separable in the elliptic coordinates on $\mathbb{R}^d$ if and only if $V(x)$ is a solution of the linear system of partial differential equations

$$(b_i - b_j)\frac{\partial^2 V}{\partial x_i \partial x_j} + \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right)\left(2V + \sum_{k=1}^{d} x_k x_k \frac{\partial V}{\partial x_k}\right) = 0, \tag{15}$$

for $i \neq j$ (see [2,30]). We shall denote the Hamiltonians of the geodesic flows of metrics $\mathrm{d}g_k^2$ and $\mathrm{d}\bar{g}_k^2$ by $H_0^k(p, x)$ and $\bar{H}_0^k(p, x)$, respectively. Suppose that $V(x)$ is a solution of (15). Then from [5] follows that Hamiltonian systems with Hamiltonian functions

$$H^k(p, x) = H_0^k(p, x) + V(x), \qquad \bar{H}^k(p, x) = \bar{H}_0^k(p, x) + V(x) \tag{16}$$

are completely integrable, and can be solved by separation of variables in elliptic coordinates for all k. There is a complete set of commuting integrals of the form

$$I_i^k(p, x) = J_i^k(p, x) + f_i(x), \quad i = 1, \dots, d,$$

for each Hamiltonian $H^k(p, x)$ where $J_i^k(p, x)$ is given by (13). The functions $f_i(x)$ do not depend of k. They are solutions of the equations $\nabla f_i(x) = S_i \nabla V(x)$, where $\nabla f = (\partial_1 f, \dots, \partial_n f)$ and S_i are given by (14). Similar statement holds for Hamiltonian systems with Hamiltonians $\bar{H}^k(p, x)$. Consider the billiard systems with Hamiltonians (16) within the ellipsoid Γ. From the choice of $I_i^k(p, x)$ we have that these functions do not change under the reflection. Thus we get the following corollary.

Corollary 2. *Suppose that $V(x)$ is a solution of* (15). *Then the billiard systems with Hamiltonians* (16) *within the ellipsoid Γ are completely integrable*.

Let us consider the solution of equations (15) in the form of Laurent polynomials

$$V(x) = \sum_{k_- \le i_1, \dots, i_d \le k_+} p_{i_1,\dots,i_d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}, \quad k_-, k_+ \in \mathbb{Z}. \tag{17}$$

Suppose that Laurent polynomial (17) is a solution of (15). Then coefficients $p_{i_1,\dots,i_d}$ satisfy the following system of difference equations:

$$\begin{aligned}&(b_k - b_l) i_k i_l p_{i_1,\dots,i_d}\\ &\quad = (i_1 + \cdots + i_d)(i_l p_{i_1,\dots,i_{k-1},i_{k-2},i_{k+1},\dots,i_d} - i_k p_{i_1,\dots,i_{l-1},i_l-2,i_{l+1},\dots,i_d}).\end{aligned} \tag{18}$$

Such potential perturbations are described for $d = 2$ in [14] (see also [15,24]) and for $d = 3$ in [17]. In general, the linear space of Laurent polynomial solution of (15) has a basis of the form

$$\mathcal{V}_k = \mathcal{V}_k(x_1, \dots, x_d), \qquad \mathcal{W}_k^i = \frac{1}{x_i^{2k}} \mathcal{P}_{k-1}^i(x_1, \dots, x_d), \quad i = 1, \dots, d, \tag{19}$$

where $\mathcal{V}_k$ and $\mathcal{P}_k^i$ are polynomials of degree $2k$, $k \geq 0$. The potentials $\mathcal{V}_k$ and $\mathcal{W}_k^i$, in elliptic coordinates, correspond to the potentials

$$V(\lambda) = \sum_{j=1}^{d} \frac{v(\lambda_j)}{\Pi_{l \neq j}(\lambda_j - \lambda_l)},$$

with $v(t) = \sum_{j=1}^{k} \alpha_j t^{d-1+j}$ and $v(t) = \sum_{j=1}^{k} \beta_j (t - a_i)^{-j}$, respectively.

Example 2. As an example, we write down a few of the basis potentials (19):

$$\mathcal{V}_1(x) = \sum_j x_j^2 \quad \text{(Jacobi)}, \qquad \mathcal{V}_2(x) = \sum_j b_j x_j^2 - \left(\sum_j x_j^2\right)^2,$$

$$\mathcal{V}_3(x) = \sum_j b_j^2 x_j^2 - 2\left(\sum_j x_j^2\right)^2 \left(\sum_j b_j x_j^2\right) + \left(\sum_j x_j^2\right)^3,$$

$$\mathcal{W}_1^i(x) = \frac{1}{x_i^2} \quad \text{(Rosochatius)}, \qquad \mathcal{W}_2^i(x) = \frac{1}{x_i^4}\left(1 + \sum_{j \neq i} \frac{x_j^2}{b_i - b_j}\right),$$

$$\mathcal{W}_3^i(x) = \frac{1}{x_i^6}\left(1 + \sum_{j \neq i}\left(\frac{2x_j^2}{b_i - b_j} + \frac{x_j^2 x_i^2}{(b_i - b_j)^2}\right) + \sum_{j,k \neq i} \frac{x_j^2 x_k^2}{(b_i - b_j)^2 (b_i - b_k)^2}\right).$$

Let $\mathcal{V}(x) = \sum_p \alpha_p \mathcal{V}_p(x)$ be some separable polynomial potential. Consider billiard systems with Hamiltonians (16). By the Maupertiues principle [1], for a given value of total energies h, satisfying condition $h > \max_{x \in D} \mathcal{V}(x)$, the motions in the potential field $\mathcal{V}(x)$ inside Γ are reduced to geodesical motions with metrics

$$(h - \mathcal{V}(x))\, \mathrm{d}g_k^2, \qquad (h - \mathcal{V}(x))\, \mathrm{d}\bar{g}_k^2. \tag{20}$$

It is clear that the billiard systems within Γ with metrics (20) are integrable.

Concluding remarks: Theorem 3 holds also if the boundary of the billiard is the union of the confocal quadrics $\Gamma = \Gamma_{c_1} \cup \Gamma_{c_2} \cup \cdots \cup \Gamma_{c_r}$, $\Gamma_{c_i} = \{x \in \mathbb{R}^d, \gamma(c_i) = 1\}$, or more generally, if the billiard is constrained to some of confocal quadrics. The same results can be formulated for billiards constrained on spheres by using geodesical equivalences established in [29,35]. Then systems are orthogonally separable in the spherical elliptic coordinates. Polynomial potentials separable in elliptic coordinates on $\mathbb{R}^d$ and spheres S^d are given by Bogoyavlenski [4] and Wojciechowski [39]. After Rosochatius's potential

$V(x) = \sum_{i=1}^{d} \alpha_i x_i^{-2}$ (see [33] and Appendix in [31]), particular examples of rational potentials are found by Braden [6] and Wojciechowski [39]. Kalnins et al. described separable rational potentials in terms of certain recurrence relations between potentials of different degrees [25]. Recently, Fedorov integrated the elliptical billiard with the elastic potential in the Euclidean space [21]. Dragović found the connection between the Laurent potential perturbations of the elliptical billiards for $d = 2$ and the Appell hypergeometric functions [16]. Also recently, the two-dimensional billiards with smooth boundary and additional irreducible integrals of the third and fourth degree, with a help of the integrable cases of Goryachev–Chaplygin and Kovalevskaya are given by Kozlova [27].

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